Lecture Notes - Sediment Transport – The flow problem

Overview

In the last lecture, we emerged with a transport model in which the rate of transport q_s depends on the shear stress τ. Recall that, for the typical range of flow in coarse-bedded streams, τ rarely exceeds the critical value τ_c by more than about a factor of two, and that, over this range, the transport model is very steep and strongly nonlinear. This places a premium on getting τ correct, because small errors in τ can lead to very large errors in estimated transport rate. Unfortunately, getting a good estimate of τ is not an easy thing. In this lecture, we will consider how to estimate τ and consider three factors that make determining τ difficult. In the fourth lecture, we will return to the issue, and suggest an alternative approach (in effect, when we use a few transport samples to calibrate a transport model, we are calibrating our estimate of τ)

There are three basic reasons why it is difficult to estimate the τ driving the transport:

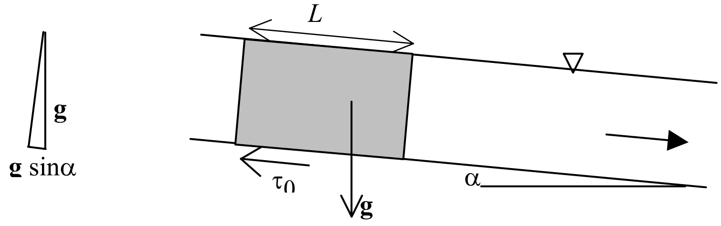
(1) unsteady and nonuniform flow: although often neglected, accelerations in the flow can have a substantial effect on τ.

(2) total stress vs. grain stress, or skin friction: although we can estimate the total force per area acting on the channel boundary; only a portion of this total stress acts on movable grains to produce transport. Determining this portion is called "drag partitioning", the methods for which are approximate.

(3) spatial variability: τ tends to vary across and along the channel. Although the total τ acting on a section can be determined, and the grain stress estimated, the nonlinear nature of the transport function means that a prediction based on the total τ can be inaccurate. There are ways to estimate local shear stress, but these require local measurements, or extensive detailed information about the channel topography and bed material.

1a. Boundary stress in steady, uniform flow

By steady, we mean that the flow is not accelerating in time (the discharge remains constant). By uniform, we mean that the slope, size, shape, and roughness of the channel remain constant along its length. If both of these are true, then the flow in the channel is not accelerating. Recalling that Newton's Second Law ($\Sigma F = ma$) states that the acceleration of a system times the mass of the system is equal to the sum of all forces acting on that system. If our system is the water in a short reach of steady, uniform flow, the system undergoes no accelerations and the forces acting on it must be balanced ($\Sigma F = 0$).



These forces are the downslope component of the weight of water in the reach and the total boundary shear force τ_0, which is resisting the flow. Equating these,

$$\rho g \sin \alpha A L = \tau_0 P L \tag{1}$$

where A is the cross-sectional area of the flow, L the length of the reach, and P is the wetted perimeter of the flow. The units of (1) are those of force and we call this a force balance. Solving (1) for τ_0, we get

$$\tau_0 = \rho g R S \tag{2}$$

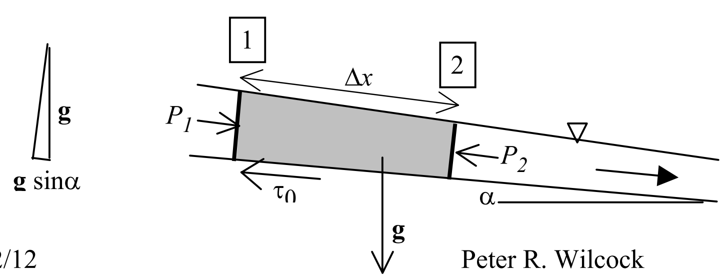
where R is the hydraulic radius, given by A/P, and S is the bed slope, given by $\sin\alpha$. (We actually use $\tan\alpha$, or rise over run, to measure bed slope, but at the slopes typical of most rivers, $\sin\alpha$ essentially equals $\tan\alpha$.)

Although (2) uses the hydraulic radius, it is often referred to as the "depth-slope product". For wide channels, depth h nearly equals R, but the radius is the correct term to use to estimate τ_0 in nonaccelerating flows.

1b. Boundary stress in unsteady, nonuniform flow

Now, we consider the more complex and realistic case in which the flow can accelerate in both time (discharge changes) and in space (flow is nonuniform). These accelerations are still balanced by the sum of the forces acting on the fluid in our reach, and Newton's Second Law still provides our governing equation. To keep things manageable, we will simplify things a bit, but not so much that we lose the essence of the problem. In particular, we will assume that the channel width b is constant and that the flow is predominantly in the downstream direction (basically we are avoiding big steps in the bed and swirly flow such as that behind obstructions and sudden changes in flow width). Neither of these assumptions is essential—the same governing equation emerges without them—but they allow for a much simpler derivation. With these simplifications, a one-dimensional model will suffice; it is called the 1d form of the St Venant equations (aka shallow water equations). First, we will simply present the equation and discuss its parts, and then we will do an abbreviated derivation, the goal of which is to reinforce the underlying physical concepts.

To visualize the problem, we will use a reach with length Δx and inclination α with $\sin\alpha = S$. Flow goes from section 1, where the depth, cross-section area, pressure, and velocity are h_1, A_1, P_1 and U_1 to section 2 with h_2, A_2, P_2, and U_2. The mean cross-section area, wetted perimeter, hydraulic radius, and velocity for the reach are A, P, R, and U, respectively.



The 1d St. Venant equation is

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = gS - g \frac{\partial h}{\partial x} - \frac{\tau_0}{\rho R} \qquad (3)$$

First, we describe the parts of the equation. The left side describes the fluid acceleration (the "ma" part of Newton's Second Law). The first term is the rate of change of velocity with respect to time (the rising limb of a flood wave, for example) and the second term describes the rate of change of velocity because the flow is accelerating from one end of our section to the other. An essential point here is that accelerations in the fluid (whether in time or in space) are associated with forces. This is what Newton's Second Law tells us! One way to visualize this is to think of flow through a nozzle on a garden hose. The nozzle accelerates the flow and it requires a force to do so (if the nozzle were not tightly screwed on to the hose, it would fly off in the direction of the flow, indicating that the nozzle is accelerating the flow by applying a force in the direction opposite to the flow). Similarly, if you gradually turn up the velcocity in the hose (by opening the valve), the force exerted on the nozzle (and by the nozzle on the fluid) will also gradually increase.

On the right side of the equation are all the relevant forces (the ΣF part of Newton's Second Law). The first is the downslope component of the weight of the water in the reach—this is what drives the flow (water flows downhill)—and we included it in our derivation of the depth-slope product. The second term is due to the pressure forces acting on the two ends of the reach. For our case, fluid pressure is essentially hydrostatic, meaning that pressure increases linearly with depth below the water surface just as in a swimming pool: pressure $P = \rho g z$ where z is depth below the surface. If the flow depth at one end of our nonuniform reach is different than at the other, then there will be an unbalanced pressure force that will play a role in accelerating the fluid in the reach. The last term represents the boundary shear stress: the force exerted on the flow by the bed and banks (recall this from our steady uniform model above). Because our interest is in the transport of sediment residing on the bed and banks, we are particularly interested in this last term.

Now, lets try to breath some life into these terms with a simple derivation, using the reach sketched below. We again write the governing equation in units of force, with accelerations on the left and forces on the right.

$$\rho A \Delta x \frac{\partial U}{\partial t} + \rho U_2^2 A_2 - \rho U_1^2 A_1 = P_1 A_1 - P_2 A_2 + \rho g A \Delta x \sin\alpha - \tau_0 P \Delta x \qquad (4)$$

Again, the first term on the left is the "unsteady" term, describing changes in the flow in time (opening the valve) and the second two terms describe the momentum of the fluid leaving and entering the reach. A change in this "momentum flux" will be associated with a force, just as in flow through a nozzle. On the right are the pressures on the upstream and downstream ends of the reach, the downslope component of the fluid weight and the boundary shear force. We need a little manipulation to get to the St Venant equation. First we substitute $Q = UA$ into the acceleration terms on the left, $P = 1/2\rho gh$ for the mean pressure on the right, $\sin\alpha \approx S$, and $A = bh$

$$\rho A \Delta x \frac{\partial U}{\partial t} + \rho Q(U_2 - U_1) = \frac{\rho g b}{2}\left(h_1^2 - h_2^2\right) + \rho g A \Delta x S - \tau_0 P \Delta x \tag{5}$$

Now we divide by $P\Delta x$ and use $R = A/P$ and $Q = UA$ again

$$\rho R \frac{\partial U}{\partial t} + \rho U R \frac{(U_2 - U_1)}{\Delta x} = \frac{\rho g b}{2P}\frac{h_1^2 - h_2^2}{\Delta x} + \rho g R S - \tau_0 \tag{6}$$

Noting that

$$\frac{\Delta h^2}{\Delta x} = 2h \frac{\Delta h}{\Delta x} \tag{7}$$

the pressure term can be rewritten as

$$\frac{\rho g b}{2P}\frac{h_1^2 - h_2^2}{\Delta x} = \frac{\rho g b h}{2P}\frac{h_1 - h_2}{\Delta x} \tag{8}$$

Using this in (6) and dividing by ρR gives

$$\frac{\partial U}{\partial t} + U \frac{U_2 - U_1}{\Delta x} = g \frac{h_1 - h_2}{\Delta x} + gS - \frac{\tau_0}{\rho R} \tag{9}$$

Finally, letting Δx shrink to infinitesimal size and using the definition of a derivative, we recover (3)

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = gS - g\frac{\partial h}{\partial x} - \frac{\tau_0}{\rho R} \tag{3}$$

The last thing we want to do is solve (3) for τ_0, since that is what will drive the transport.

$$\tau_0 = \rho g R\left(S - \frac{\partial h}{\partial x} - \frac{U}{g}\frac{\partial U}{\partial x} - \frac{1}{g}\frac{\partial U}{\partial t} \right) \tag{10}$$

One way to interpret this result starts from the fact that, if the flow were steady and uniform (meaning that all the derivatives in (10) would be equal to zero), we recover our depth-slope product in (2). If the flow is unsteady and/or nonuniform, the other three terms on the right of (10) come into play. The question is, how big are these three terms compared to *S?* Here, we arrive at a very important point. You could *assume* that these derivative terms are small. This is sometimes true and sometimes incorrect. *How would you know?* It is far preferable to evaluate the magnitude of these terms and determine explicitly whether neglecting the derivative terms is a reasonable thing to do. The first case is an unsupported *assumption*; the second is based on an *approximation*, which allows a demonstration of whether the assumption is a reasonable one. The attached spreadsheet "NonuniformFlow.xls" illustrates the problem for the simple, but nontrivial case of steady flow that changes depth in the downstream direction. The spreadsheet "Flood.xls" illustrates the interesting case in which a tributary flood causes the mainstem discharge to rapidly increase over a defined period of time, producing a wave of water in the mainstem.

2. The drag partition

So far, we have discussed how to estimate the total boundary stress τ_0 in a stream reach. This gives us the *total* force acting on the wetted boundary of bed and banks. Some of this force acts on the movable grains on the stream bed and thus drives the transport, but some of it also acts on other things: woody and other debris in the channel, bridge piers, channel bends, etc. To estimate the sediment transport rate, we need to *partition* total stress τ_0 into that part that acts only on the sediment grains. We'll call this the grain stress τ' (this is also called the skin friction). We have no direct way to estimate τ', although there are some useful approximate approaches. We will develop one approach here, based on the Manning Equation

$$U = \frac{\sqrt{S} R^{2/3}}{n} \tag{11}$$

where *n* is the Manning roughness. Typical values of *n* for natural streams are in the range 0.03 to 0.08, although larger values are observed for very rough or very bendy channels, particularly when they are clogged with vegetation.

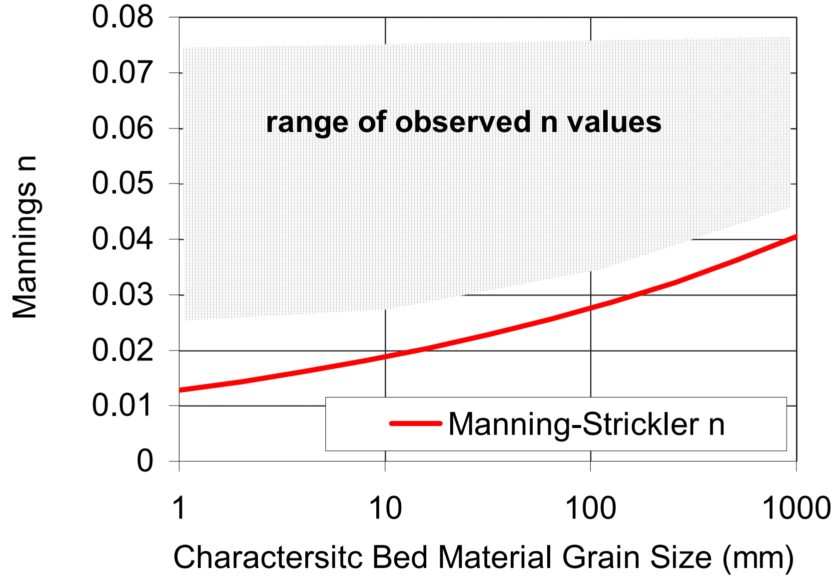
A number of factors contribute to the boundary roughness and, therefore, to the magnitude of *n*. One of these (the one we are interested in) is the bed grain size. You might reason (correctly) that larger grains would be hydraulically rougher than smaller grains. By (11), that means that, for the same *U* and *S*, a bed with coarser sediment with have a larger depth. An approximate relation between *n* and a characteristic grain size of the bed material, often referred to as the Strickler relation, is

$$n = 0.040 D^{1/6} \tag{12}$$

for *D* in m, or

$$n = 0.013 D^{1/6} \tag{13}$$

for *D* in mm.



Notice that Manning's equation contains both *R* and *S*, suggesting that we might be able to pull τ_0 out of the deal via the depth-slope product (in fact, that is just what flow resistance equations are all about: a relation between velocity, flow geometry, boundary roughness, and τ_0). If we multiply (11) by $(\rho g)^{2/3}S^{1/6}$ and rearrange, we get

$$(\rho g)^{2/3} S^{1/6} nU = (\rho g RS)^{2/3} \tag{14}$$

Raising all this to the 3/2 power gives

$$\rho g S^{1/4} (nU)^{3/2} = \tau_0 \tag{15}$$

Now here is the tricky part. Suppose we insert the Strickler definition of *n* into (15). Recalling that other factors also contribute to *n*, the Strickler *n* should be smaller than the total *n* for the channel. By using the Strickler *n* in (15), we are essentially calculating the shear stress due to the bed grains only, which is the approximation of τ' that we are after. Using (13) in (15), we get

$$\rho g (0.013)^{3/2} (SD)^{1/4} U^{3/2} = \tau' \tag{16}$$

Now, we have to face up to the choice of a characteristic grain size *D*. We haven't discussed the role of bed grain size in roughness yet, but hopefully it makes sense that it would be the larger sizes in the bed that would tend to dominate the roughness. For example, D_{90} and D_{84} are often used (these are the grain sizes for which 90% or 84% of the bed material is finer). We will use $2D_{65}$, based on field and lab observations, although it is difficult to make a strong case for any particular value of *D*. Fortunately, the choice ends up not making a big difference (because *D* is found in (16) raised to the power ¼). Substituting $D=2D_{65}$ in (16) and using ρ = 1000 kg/m^3 and g = 9.81 m/s^2, we get

$$\tau' = 17(SD_{65})^{1/4} U^{3/2} \quad (17)$$

for τ' in Pa, D_{65} in mm, and U in m/s. We see that τ' depends mostly on the flow velocity (meaning that it depends on Q and all the factors—channel size, shape, slope—that relate Q and U) and, to a lesser extent, on S and D_{65}.

3. Spatial variability

Recall that τ rarely exceeds τ_c by more than about a factor of two in most coarse bedded streams. This means that our transport function (e.g. Meyer-Peter & Muller) will be very steep and nonlinear for most flows for which we might wish to estimate transport rate. Recall, also, that the local shear stress τ_l can also vary considerably across and along a stream reach. This is a problem. If the transport function were linear, we could calculate an average value of τ' and then use that in our transport formula. Because the transport function is strongly nonlinear, however, this will produce errors that can be significant. Basically, the rate of transport in areas where τ' is greater than the mean will be much larger than the rate transport in areas where τ' is smaller than the mean. A simple case would be one in which the mean τ' is less than τ_c, indicating that no transport should occur. But even if the mean $\tau' < \tau_c$, there can still be locations where $\tau_l > \tau_c$. Thus, the mean τ' indicates no transport when, in fact, there will be transport going on! The attached spreadsheet "variable stresses.xls" calculates transport rates using mean τ' and using an estimate of τ_l. The case is for a simple (but illustrative) channel section divided into two equal regions of different depth.

The steep nonlinear transport function, combined with the spatial variability in both τ and D indicate that transport rate is really a local kind of thing. At a given flow, some places on the bed might have little or no transport going on while other places may have very active transport. So, if you want to fully understand the transport, you will need to measure τ_l. In a research context, you might be interested in measuring both τ_l and q_s across a channel section. A practical interest in τ_l will arise when if wish to know the transport rate, or perhaps just the likelihood of entrainment, over particular locations on the bed, such as over salmonid redds.

We cannot measure τ_l directly and, instead, use measurements of velocity in the flow directly above the point of interest on the bed. Generally, we take advantage of the fact that the flow velocity (here we are talking about the velocity at a point, which we will designate u in order to distinguish it from the section-averaged velocity U that we have already been discussing) varies logarithmically with height z above the bed. We represent this as

$$\frac{u}{u_*} = 2.5 \ln\left(\frac{z}{z_0}\right) \quad (18)$$

where $u_* = (\tau/\rho)^{1/2}$ is called the shear velocity and z_0 is a bed roughness length that corresponds to the elevation where u goes to zero. Field observations indicate that $z_0 \approx 0.1D_{90}$. It turns out the constant 2.5 is quite general and holds for a wide range of turbulent flows, including pipe flow as well as open-channel flow. Strictly speaking, (18) only holds in a region of the flow that is *both* somewhat above the bed surface (say, $z > 3D_{90}$) and well below the water surface (say, $z < 0.2h$). This indicates that flows that are shallow compared to their bed material ($h/D_{90} < 15$ by the rules just given) will have no "log layer". But, for flows that are deeper than this and are approximately steady and uniform, it turns out that (18) applies reasonably well throughout the entire flow depth. This motivates consideration of the depth-averaged version of (18)

$$\frac{U_l}{u_*} = 2.5 \ln\left(\frac{h}{ez_0}\right) \tag{19}$$

where U_l is the depth-averaged velocity, using the subscript l to distinguish it from the section-averaged velocity U, and e is the base of the natural logarithms.

We can use (18) or (19) to estimate u_* from velocity observations. The most common approach is to measure u at a number of different elevations above the bed. We then fit a straight line to the observed u as a function of $\ln(z)$. If we rewrite (18) as

$$u = 2.5u_* \ln(z) - 2.5u_* \ln(z_0) \tag{20}$$

we see that the slope of this line $\sigma = 2.5u_*$. Thus, $\tau_l = \rho(\sigma/2.5)^2$ (using the definition of u_*). We could also estimate u_* by making a single observation of u at some near-bed z. In this case, we have to provide an independent estimate of z_0 (e.g. $z_0 = 0.1D_{90}$). Finally, we could determine U_l from a number of (u, z) observations and, with an independent estimate z_0, estimate u_* using (19). Wilcock (*WRR*, 1996) used replicate field measurements to evaluate the precision of each of these approaches. The velocity profile approach offered the least precision of the three methods, although it has the important advantage that an independent estimate z_0 is not required. The estimate based on (19) offered the highest precision (because U_l and h can be measured more accurately than the value of u at any particular z), but requires an independent estimate of z_0 as well as the assumption that (18) holds throughout the flow depth (although the latter assumption can be evaluated based on the observations used to determine U_l). Using a single near-bed observation of (u, z) offers precision intermediate to the other methods. Although requiring an independent estimate z_0, this last method depends on the existence of a log layer only very close to the bed, making it applicable to a wider range of flows than the other two methods. Because it requires only a single measurement, it can be done quickly.

Modern acoustic velocity probes, which use the Doppler effect to determine velocities throughout the flow, offer the possibility of determining τ_l in an entire reach with far less effort than that required to determine local velocities with traditional current meters. These newer probes generally require flow depths larger than those found in many coarse-bedded streams, however, and, because they are expensive and fragile, are rarely used in flows producing large transport rates in gravel-bed rivers.

An alternative approach is to estimate τ_l it using a flow model. If the topography and roughness throughout a reach is known, a depth-averaged two-dimensional flow model (similar to eqn (3) with additional terms for the lateral variation in U and *h* and cross-stream velocity *V*) can provide accurate estimates of τ_l throughout the reach. Unfortunately, the necessary detail of topography and roughness throughout a reach is difficult to collect and rarely available.

4. Transport Model Based on Q: Part II

Recall that a model using Q to predict transport rates would be most useful, since many transport problems are ultimately defines in terms of *Q* and Q_s. Lets return to this problem, now that we have an idea of how to determine the grain stress τ'. The idea is to evaluate if a transport model in the form

$$\frac{Q_s}{Q_{sr}} = \left(\frac{Q}{Q_r}\right)^{\beta} \tag{21}$$

can be general.

To start, lets use a simplified transport model

$$q^* \propto \left(\tau^*\right)^3 \tag{22}$$

which approximates the transport rates over the typical range of interest in gravel-bed streams. {(22) is known as the Einstein-Brown model) For our purposes, all we need is

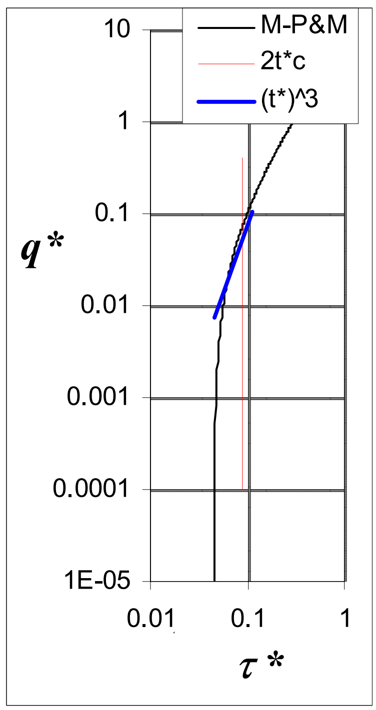
$$q_s \propto \tau'^3 \tag{23}$$

and, forming a ratio with the same relation defined at the reference value we get

$$\frac{q_s}{q_{sr}} = \left(\frac{\tau'}{\tau'_r}\right)^3 \tag{24}$$

Now, we need to relate τ' to Q. We get part of the way there using our Manning-Strickler drag partition

$$\tau' = 17(SD_{65})^{1/4} U^{3/2} \tag{25}$$



Writing (25) a second time for τ' at some reference level τ'_r, and forming a ratio between the two, we get

$$\frac{\tau'}{\tau_r'} = \left(\frac{U}{U_r}\right)^{3/2} \tag{26}$$

Inserting this expression into (24), we get

$$\frac{q_s}{q_{sr}} = \left(\frac{U}{U_r}\right)^{4.5} \tag{27}$$

So we see that the ratio of transport at two different velocities in a reach may be expressed as a constant power of those two velocities, assuming that the slope and bed grain size do not change between the two flows (otherwise S and D_{65} would not cancel in going from (25) to (26). This also assumes that the width of the portion of the channel producing bed-material transport remains constant as the flow changes, which often is not a bad assumption.

Now, we have to relate U to Q. Fortunately, there have been many measurements of this, which can be summed up in terms of the **hydraulic geometry**. There are actually two forms of the hydraulic geometry, one for a cross section ("at-a-station hydraulic geometry") and another for different sections and different rivers ("downstream hydraulic geometry"). Here, we use the at-a-station form. In short, the idea is that the variation of channel width B, depth h, and mean velocity U are assumed to follow power relations with Q:

$$B = aQ^b$$
$$h = cQ^f \tag{28 a-c}$$
$$U = kQ^m$$

Because $Q = BhU$, we know that

$$ack = 1 \text{ and } b + f + m = 1 \tag{29}$$

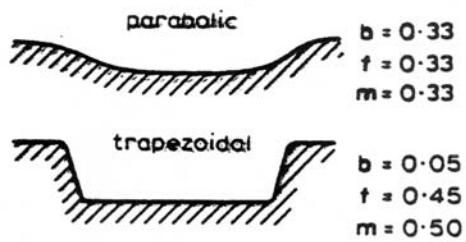
The plots below show values of the exponents b, f and m for parabolic and trapezoidal channels and their range for many channels in a range of environments. Because $b + f + m = 1$, these results can be plotted accurately in a ternary diagram.

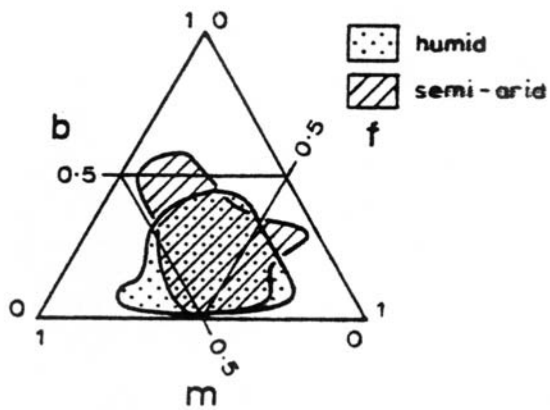
It is the last of the three hydraulic geometry relations (28c) that we need to finish our problem. Inserting (28-c) into (27), we get

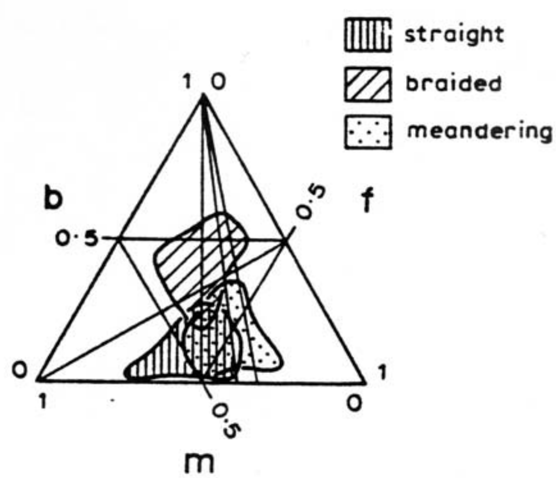
$$\frac{q_s}{q_{sr}} = \left(\frac{Q}{Q_r}\right)^{4.5m} \tag{30}$$

The compiled record suggests that the exponent m varies between about 0.3 and 0.8, meaning that our exponent β varies between 1.4 and 3.6. Higher values of β are also observed, particular in steep mountain streams, which can have very small transport rates, such that the exponent 3 in the Einstein-Brown formula underestimates the actual variation of q^* with τ^*.

We will return to (28-c) in the fourth lecture. There are relatively simple ways to determine the exponent *m* for a channel, and we will exploit this in developing a simple method of estimating sediment transport rates.







5. Summary

For the purposes of estimating transport rates, we can estimate the total boundary stress τ_0, or we can estimate the local boundary stress τ_l. With the former, we must perform a drag partition to determine the grain stress τ', the portion of τ_0 acting on the grains and driving the transport. Because τ_l varies across the bed, the transport rate calculated using a mean value of τ' will not equal the sum of the local transport rates calculated using τ_l. Nonetheless, mean τ' may provide as a useful index of flow strength for the purpose of estimating transport rates. Estimates of τ_l are appropriate when we are interested in transport at a particular location (e.g. over spawning gravels). If we want to estimate the total transport rate through a section, we would have to measure τ_l throughout the section or find a way to index τ_l to τ'.

Take-Home Tools

The St.Venant equation (3) provides a basis for determining whether the total boundary stress τ_0 can be estimated using the depth-slope product ρgRS.

To determine transport rates, the grain stress τ' portion of τ_0 must be estimated. Equation (17) provides a simple means of estimating τ'.

Local stresses can be estimated from velocity measurements at particular stations above the bed. Three different methods provide a tradeoff between precision and the range of applicable conditions.